

Currency Cross Rate and Triangular Arbitrage in Nigerian Exchange Market

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Abstract— Economic and financial phenomena are analyzed by models based on statistical physics; for example modeling of financial markets as stochastic processes. Stochastic models of stock price changes have been studied by many authors, only that these models discuss change in one price. This paper studies the existence of interaction between the prices of the markets caused by a financial activity called currency cross rate and currency cycle (or triangular arbitrage transaction) as in the Nigerian exchange markets. A simple economic model where the time evolution is described by an equation capturing the currency cross rate under arbitrage is studied. We show that the mean-field limit of this equation as well as the wealth of the investors (the arbitrageurs) in this market is distributed according to the power-law type.

Index Terms— Currency cross rate, Power-law distribution, Nigerian exchange market, Triangular arbitrage.

I. INTRODUCTION

Like almost anything else, the value of any currency is determined by supply and demand. The greater the demand in relation to the supply, the greater the value, and vice versa. This is true even when no foreign exchange is involved. For instance, if a country never expands its money supply, then the money that is available becomes more valuable as the economy expands. Thus, the price of individual items decreases, which is **deflation**.

When the money supply expands faster than the economy, then money becomes less valuable. It takes more money to purchase single items, leading to **inflation**.

Naturally, in foreign exchange, when currency of a particular country is plentiful, it will have less value against other currencies, and vice versa.

Demand for the currency also affects the exchange rate. If a country pays a significantly higher interest rate than another country, or has significantly more investment opportunities or a more stable government, then that country's currency will have greater value than that of the other country.

Cross rates are the exchange rates of a currency with other currencies, and those currencies with each other. Cross rates are equalized among all currencies through a process called triangular arbitrage. Below is a table of key cross rates of some major currencies.

TABLE 1: CROSS RATE OF SOME MAJOR CURRENCIES.

Key Cross Rates for 5/14/2007.
(SourceMarketWatch.com)

Currency	USD	GBP	EUR	JPY
USD		0.50443	0.73951	120.10500
GBP	1.98244		1.46603	238.10043
EUR	1.35225	0.68211		162.41160
JPY	0.83260	0.41999	0.61572	

From this table, we can calculate what the USD is worth in EUR, thus;

$$\text{EUR/USD} = \frac{1.98244 \text{ USD}}{1 \text{ GBP}} \times \frac{.68211 \text{ GBP}}{1 \text{ EUR}} = 1.3522421,$$

Which is very close to the actual quote, 1.35225, from the table above. Because we are ignoring the bid/ask spread and transaction costs to simplify the mathematics in this example, there is no reason to believe that it would be exact. It is also true that arbitrage is not a perfect equalizer because the market is not perfectly efficient.

Assuming that exchange rate from 1 NGN-to- USD is 2, one would expect that the rate from USD-to-NGN = $\frac{1}{2}$.

Suppose for a second that it was not, but that the rate from USD-to-NGN is 0.6. An investor quickly buys 10 NGN with 5 USD. He could then take his 10 USD and exchange them back for NGN and get back 6 NGN. In this case, he has one more NGN than before. This type of exchange is known as **arbitrage**.

Since the investor gained a NGN, and since we are not creating or destroying any currency, the rest of the market must have lost a NGN. This of course is bad for the rest of the market. We would expect that the other agents in the currency exchange market will change the exchange rates that they offer so that these opportunities to get exploited are taken away. Still there is a class of investors known as **arbitrageurs** who try to exploit these differences.

Arbitrage in general takes on more complex forms than this, involving several currencies. Suppose that the NGN-to-USD exchange rate is 2 and the USD-to-GBP is 3. To figure out what the NGN-to-GBP exchange rate needs to be, we just multiply the two rates together:

$$\text{NGN} - \text{to} - \text{GBP} = (\text{NGN} - \text{to} - \text{USD})(\text{USD} - \text{to} - \text{GBP}). \quad (1)$$

Equation (1) is an exchange rate's property known as **transitivity**. To avoid arbitrage we would need the NGN-to-GBP exchange rate to be 6 and the GBP-to-NGN

exchange rate to be $\frac{1}{6}$. If it were $\frac{1}{5}$, then the investor could again make an additional NGN and the rest of the market would lose one. The arbitrageur in this market transacts whenever he recognizes this opportunities. He makes profit if it takes him a few seconds to recognize the currency circle opportunities and executes the transaction (in this market, the rate from NGN-to-USD is never the same as from USD-to-NGN, etc).

For any three currencies A, B, and C trading A for B, B for C and C for A is known as currency circle. A financial activity that takes advantage of three exchange rates among three currencies is known as triangular arbitrage. The triangular arbitrage makes the product of the three exchange rates converge to a certain value (Moosa, 2001 [2]). Aiba, et al, (2002 [1]) had shown that there is in fact triangular arbitrage opportunities in foreign exchange markets and that they generate an interaction among foreign exchange rates.

In the Nigeria exchange market, an agent in the currency market cannot exchange a JPY- to- USD, USD -to- GBP and GBP-to-JPY without making a stop to the NGN, as shown in figure 1 below. This special case of financial activity in the Nigeria exchange market (with Aba as case study) is studied herein. The rest of the paper is organized as follows; the next section establishes the currency cross rate problem. In section three, an analytical approach is used to model a mean-field limit stochastic differential equation which captures the exchange rate. The wealth of the investor is shown to be a power-law type in section four. Section five concludes.

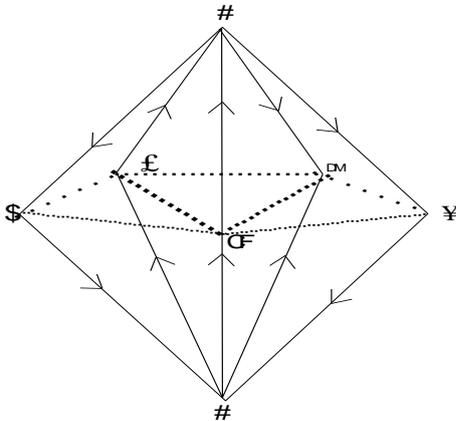


Figure 1: Graphical illustration self-closing triangle of currency exchanges called **triangular arbitrage**. The dotted lines show currency cross rate of more than three currencies. This implies that there exist interactions between more than three currencies in the foreign exchange market.

II. THE CURRENCY CROSS RATE PROBLEM

There are n types of currencies, and at each exchange one currency is picked at random. Table 2 shows the rate of exchange in NGN in seconds to various currencies, taken from Aba exchange market Abia State, Nigeria, November 2010. How many exchanges one has to make before picking all the currencies? Let m be the number of exchanges made. We bound the probability that m exceeds a certain number and yet not all the currencies are exchanged. Let $C_i \in \{1, \dots, n\}$ be the currency exchanged at $i - th$ time.

The $j - th$ time is a success, if C_i was not exchanged before at the first $j - 1$ time. Figure 3 shows the rate of exchange from NGN to various currencies.

TABLE 2: THE RATE OF EXCHANGE IN SECONDS FROM NGN TO VARIOUS CURRENCIES (SOURCE: ABA FOREIGN EXCHANGE MARKET).

Currency	Rate 1sec	Rate 2secs.	Rate 3secs.	Rate 4secs	Rate 5secs.
USD	146	145.50	145	144	151
GBP	250	248.5	248	243.50	245
EUR	203	210	208.50	208	205
WFA	270	271.50	274	271	270
CFA	248	249.50	251.50	253	256

The exchange rate over an interval of time period $\tau = t + \Delta t$, $\mu(t)$ is given by

$$(1 + \mu(t)) = \prod_{i=1}^n (1 + \eta_i(t)). \quad (2)$$

For a short time periods (up to a month, say), the one-period exchange rates $\eta_i(t)$ are small, and $\mu(t)$ can be approximated by:

$$\mu(t) \cong \sum_{i=1}^n \eta_i(t), \quad (3)$$

with

$$v = \ln \mu(t) = \sum_{i=1}^n \ln \eta_i(t), \quad (4)$$

If one focuses on the logarithmic rate-change ratio. (Figure 2 shows the probability of the rate product). $\eta_i(t)$ is geometrically distributed with

$$E[\eta_i(t)] = n \ln n + O(n) \quad (5)$$

and

$$\lim_{n \rightarrow \infty} \frac{\text{var}[\eta_i(t)]}{n^2} = \frac{\pi}{6}. \quad (6)$$

From USD-to-GBP, one has to get to NGN and vice-versa (as in figure 1 above), that is two edges are created at each step. Thus the total number of edge creations (over all steps) is at most $2n$. Let C_{ij} denote the amount of currencies that agent j spends exchanging the currency of agent i (and vice-versa). For n types of currencies and m exchanged, there is a relationship between m and the probability of having exchanged at least one currency of the n types. The probability of having exchanged all n currency types abruptly changes from nearly zero to almost one in a small interval centered around $n \ln n$ (Motwani and Raghavan, 1995 [4]). This probabilistic quantity provides some insight into the true behavior of those quantities.

Theorem 1: Let $\eta_i(t)$ be the exchange for each of the n type's currencies. Then, for any constant $k \in \mathbb{R}$ and $m = n \ln n + kn$, $\lim_{n \rightarrow \infty} \text{Pr}[\eta_i(t) > m] = 1 - e^{-e^{-k}}$.

Proof: See proof theorem 3.8 of Motwani and Raghavan, 1995 [4].

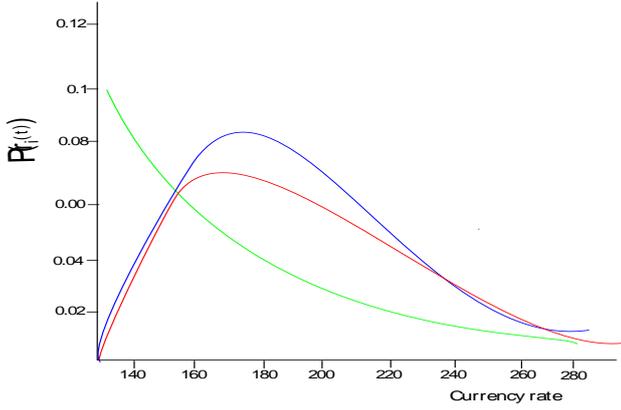


Figure 2 : The probability density function of $r_i(t)$. Green for one rate, Blue for two rates and Red for three rates.

III. ANALYTICAL APPROACH

The exchange rate of currency i at time t is defined by $r_i(t)$, (time is counted for trading seconds in multiples of a fundamental unit, seconds, say). We define continuous exchange rate as:

$$\ln r_i(t + \Delta t) - \ln r_i(t) = \ln \left(\frac{r_i(t + \Delta t)}{r_i(t)} \right), \quad (7)$$

and the discrete exchange rate as

$$\frac{r_i(t + \Delta t) - r_i(t)}{r_i(t)} = \frac{\Delta r_i(t)}{r_i(t)}. \quad (8)$$

As for the log- rates, since we are considering short time period, rates of exchange are small and log-rates are very closely approximated by:

$$\begin{aligned} \ln \left(\frac{r_i(t + \Delta t)}{r_i(t)} \right) &= \ln \left(1 + \left(\frac{r_i(t + \Delta t)}{r_i(t)} \right) \right) \\ &\cong \left(\frac{r_i(t + \Delta t)}{r_i(t)} \right). \end{aligned} \quad (9)$$

From time t to $t + \Delta t$ the exchange rate is characterized by fluctuation, so that

$$\frac{\Delta r_i(t)}{r_i(t)} = a(t) \Delta t, \quad (10)$$

due to some environmental effects. Let $a(t) = f_i(t) + \text{noise}$ as in Osu, (2010 [5]), where $f_i(t)$ is an independent fluctuation (a Gaussian random variable with mean m and variance σ^2). Then

$$\frac{\Delta r_i(t)}{r_i(t)} = (f_i(t) + \bar{\xi}_{t,\Delta t}) \Delta t. \quad (11)$$

Suppose $\bar{\xi}_{t,\Delta t} \rightarrow N(0, \Delta t)$ and $\bar{\xi}_{t,\Delta t} = g(\mu(t)) \Delta t$ (where $g(\mu)$ is an attractive interaction between currencies), then as $\Delta t \rightarrow 0$,

$$\frac{dr_i(t)}{dt} = (f_i(t) + g(\mu(t))) r_i(t). \quad (12)$$

We define $g(\mu)$ as (as a linear approximation)

$$g(\mu) = \left(\sum_{i \neq j} C_{ij} - \varepsilon \right), \quad (13)$$

so that (12) gives

$$\frac{dr_i(t)}{dt} = f_i(t) r_i(t) + C(\bar{r}_i(t)) - r_i(t). \quad (14)$$

All agents exchange with all others at the same rate, that is $C_{ij} \equiv C/N \forall i \neq j$. N here is the total number of agents (arbitrageur), and C/N , the scaling needed to make the limit $N \rightarrow \infty$ well defined and $r_i \rightarrow \vartheta r_i$ remains invariant under transformation since the unit of currency is arbitrary when all r_i 's are multiplied by common arbitrary factor (Bouchaud and Mezard, 2000 [2]). $\bar{r}_i(t) = N^{-1} \sum_i r_i$ is the average overall exchange rate. All the agents in this market feel the same influence of their environment, therefore this linear equation (14) is a 'mean-field' model and integrating this equation and summing over i gives:

$$\bar{r}(t) = \bar{r}(0) e^{(m + \frac{1}{2}\sigma^2)t}, \quad (15)$$

Which is deterministic in the limit $N \rightarrow \infty$. Equation (14) can be rewritten in terms of the normalized exchange rate $\mathfrak{S}_i = r_i / \bar{r}$ as:

$$\frac{d\mathfrak{S}_i}{dt} = \left(f_i(t) - m - \frac{1}{2}\sigma^2 \right) \mathfrak{S}_i + C(1 - \mathfrak{S}_i), \quad (16)$$

Which we can associate the following Fokker-Planck equation for the evolution of the density of exchange rate $\varphi(\mathfrak{S}, t)$ such that:

$$\frac{\partial \varphi}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2}{\partial \mathfrak{S}^2} \mathfrak{S}^2 \varphi + \frac{\partial \varphi}{\partial \mathfrak{S}} \left[C(\mathfrak{S} - 1) + \sigma^2 / 2 \mathfrak{S} \right]. \quad (17)$$

It is easy to see that the Pareto distribution $\varphi = \mathbb{F} \mathfrak{S}^{-(1+\alpha)} e^{-\left(\frac{\alpha-1}{\mathfrak{S}}\right)}$ solves (17) with

$$\alpha = 1 + \frac{C}{\sigma^2/2}, \quad (18)$$

where $\mathbb{F} = \frac{(\alpha-1)^\alpha}{\Gamma(\alpha)}$ is the normalisation factor.

IV. THE DISTRIBUTION OF THE WEALTH OF THE ARBITRAGEUR.

In this sequel, we drive the distribution for the wealth of the arbitrageur thus;

Theorem 2:

Let $r(t)$ the exchange rate from one currency to another and let $W(t)$ the wealth of an arbitrageur. Then $W(t)$ is said to be distributed according to power-law, if $r(t)$ is stochastically proportional to $W(t)$.

Proof.

Let the relationship between $r(t)$ and $W(t)$ be given as;

$$r(t) = \theta(t) W(t), \quad (19)$$

where $\theta(t)$ is a stochastic proportionality factor which is uncorrelated with $W(t)$, whose probability density function is given by $\rho(\theta)$, say. We assume here that exchange rate is always positive ($r > 0$). Let

$$P(r: r > 0) = G(r) = \int_{-\infty}^{\infty} \rho(\theta) H(W) d\theta, \quad (20)$$

where $G(r)$ and $H(W)$ are the cumulative distributions of r and W respectively. Thus the density function of r is given by

$$g(r) = \int_{-\infty}^{\infty} \rho(\theta) h\left(\frac{r}{\theta}\right) \frac{1}{\theta} d\theta, \quad r > 0. \quad (21)$$

Since the Pareto density function $h(W)$ is non-zero only for $W > W_0$, the contribution to the integral in (21) is only for values of θ such that $\frac{r}{\theta} > W_0$. Then $0 < \theta < \frac{r}{W_0}$. If $\theta < 0$ then $\frac{r}{\theta} < 0$ and $h\left(\frac{r}{\theta}\right) = 0$; if $\theta > \frac{r}{W_0}$ then $\frac{r}{\theta} < W_0$ and again $h\left(\frac{r}{\theta}\right) = 0$. Hence, we write equation (21) as;

$$g(r) = \int_0^{\frac{r}{W_0}} \rho(\theta) h\left(\frac{r}{\theta}\right) \frac{1}{\theta} d\theta, \quad r > 0. \quad (22)$$

Employing the Pareto distribution $\varphi = F\mathfrak{I}^{-(1+\alpha)} e^{-\left(\frac{\alpha-1}{\mathfrak{I}}\right)}$ for h , we have:

$$g(r) = F\mathfrak{I}^{-(1+\alpha)} e^{-\left(\frac{\alpha-1}{\mathfrak{I}}\right)} \int_0^{\frac{r}{W_0}} \rho(\theta) h\left(\frac{r}{\theta}\right)^{-\alpha} d\theta. \quad (23)$$

$g(r)$ has important property for being asymptotically Paretian; that is as $r \rightarrow \infty$, $g(r) \rightarrow F\mathfrak{I}^{-(1+\alpha)} e^{-\left(\frac{\alpha-1}{\mathfrak{I}}\right)}$, where $F = \mathfrak{I} \int_0^{\infty} \rho(\theta) h\left(\frac{1}{\theta}\right)^{-\alpha} d\theta$. Thus, the single exchange rate distribution generated by market in which the investor's wealth is distributed according to the Pareto law with exponent α is asymptotically Paretian with the same exponent α .

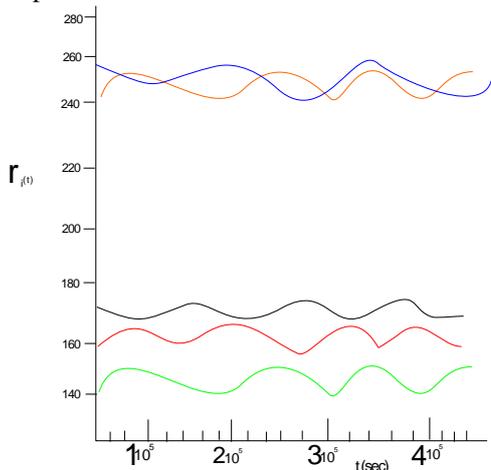


Figure 3: The rate of change in seconds from NGN to various currencies. Green; USD-to-NGN, Red; EUR-to-NGN, Black; CFA-to-NGN, Orange; WFA-to-NGN and Blue; GBP-to-NGN.

V. CONCLUSION

We first establish that the currency circle exists in the Aba foreign exchange market with the arbitrageurs making profit whenever there is opportunity and they transact. We further discussed a simple equation capturing the exchange rate between currencies. The mean-field limit of this equation was investigated and shown that the distribution of wealth of arbitrageurs is Paretian.

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